

SPECIAL BASES FOR THE VECTOR SPACE OF SQUARE MATRICES

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ABSTRACT. We describe families of complete orthogonal bases of full rank matrices which span the vector spaces of square matrices. The proposed bases generalise non-trivially the Pauli matrix while shedding light on their algebraic properties. Finally we introduce the notion k -pseudo-closure for orthogonal bases spanning vector subspaces of square matrices and discuss their connections with hadamard matrices.

1. INTRODUCTION

It is a common practice when manipulating $n \times n$ matrices to express them as linear combinations of basis elements from the complete orthonormal basis

$$B_n = \{\mathbf{e}_i \mathbf{e}_j^t\}_{0 \leq i, j < n}$$

where the set $\{\mathbf{e}_i\}_{0 \leq i < n}$ denotes the canonical euclidean basis of n -dimensional column vectors. Although a prevalent choice as canonical basis for expressing square matrices, we observe that the basis B_n is not as well behaved with respect to the matrix product operation, more precisely

$$(\mathbf{e}_{i_1} \mathbf{e}_{j_1}^t) (\mathbf{e}_{i_2} \mathbf{e}_{j_2}^t) = \begin{cases} \mathbf{e}_{i_1} \mathbf{e}_{j_2}^t & \text{if } j_1 = i_2 \\ \mathbf{0} & \text{otherwise} \end{cases},$$

in addition elements of B_n are rank 1 matrices. An important insight conferred by linear algebra is the fact that the choice of a basis for a vector spaces is crucial to the investigation of properties of a particular set of vectors. Incidentally we discuss here alternative complete orthogonal bases of full rank matrices which in some instances are also pseudo-closed under matrix product as defined bellow.

Definition 1. [Pseudo-closure] Let $k \geq 1$ and G denote an orthogonal basis for a subspace of square matrices. G is said to be pseudo-closed of order k with respect to the matrix product operation or simply a k -pseudo-closed basis if

$$(1.1) \quad \forall \mathbf{A}, \mathbf{B} \in G, \begin{cases} \mathbf{A}^{\star^{k+1}} = \mathbf{A} \\ \mathbf{B}^{\star^{k+1}} = \mathbf{B} \end{cases}, \quad \exists \mathbf{D} \text{ a diagonal matrix s.t. } \mathbf{D} (\mathbf{A} \mathbf{B}) \in G \text{ and } \mathbf{D}^k = \mathbf{I}.$$

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where for $\mathbf{A} = (a_{i,j})_{0 \leq i,j < n}$ we define $\mathbf{A}^{\star^{k+1}} := (a_{i,j}^{k+1})_{0 \leq i,j < n}$.

2. ORTHONORMAL BASIS INDUCED BY THE UNITARY GROUP

Let $\{\mathbf{e}_i\}_{0 \leq i < n}$ denote the canonical euclidean basis of column vectors. We define the *fourier complete orthogonal basis* to be the set F_n ,

$$(2.1) \quad F_n := \left\{ \mathbf{B}(k, l) = \sum_{0 \leq j < n} \left(\mathbf{e}_j \mathbf{e}_{\{j+k \bmod n\}}^t \right) \exp \left\{ i \frac{2\pi}{n} j l \right\} \right\}_{0 \leq k, l < n}.$$

We think of the set of matrices as an inner-product space with the inner-product being defined as follows

$$\langle \mathbf{A}, \mathbf{M} \rangle := \text{Tr} \{ \mathbf{A} \mathbf{M}^\dagger \}$$

hence

$$(2.2) \quad \forall \mathbf{A} \in \mathbb{C}^{n \times n}, \quad \mathbf{A} = \sum_{0 \leq k, l < n} n^{-1} \langle \mathbf{A}, \mathbf{B}(k, l) \rangle \mathbf{B}(k, l).$$

The proposed fourier basis generalizes the Pauli Matrices. In contrast to the conventional canonical orthonormal matrix basis B_n , the basis elements of the fourier basis are each full rank and it follows from their definition that they constitute a n -pseudo-closed complete orthogonal basis spanning the vector space of $n \times n$ matrices. It also follows from the definition of F_n that its elements generate a finite multiplicative group of matrices of order bounded by n^{n+1} . Furthermore the group generated by F_n is isomorphic to a subgroup of S_{n^2} which we represent here using matrices in $\{0, 1\}^{n^2 \times n^2}$. Let T denote the group Isomorphism

$$T : \mathbb{C}^{n \times n} \rightarrow \mathbb{C}^{n^2 \times n^2}$$

(2.3)

$$T(\mathbf{B}(k, l)) = \sum_{0 \leq u < n} \left(\mathbf{e}_u \mathbf{e}_{\{u+k \bmod n\}}^t \right) \otimes \left(\sum_{0 \leq v < n} \left(\mathbf{e}_v \mathbf{e}_{\{v+j \times l \bmod n\}}^t \right) \right) = \mathbf{F}(k, l),$$

the isomorphism is naturally extended to the set of all $n \times n$ matrices as follows

$$(2.4) \quad \forall \mathbf{A} \in \mathbb{C}^{n \times n}, \quad T(\mathbf{A}) = \sum_{0 \leq k, l < n} n^{-1} \langle \mathbf{A}, \mathbf{B}(k, l) \rangle \mathbf{F}(k, l).$$

so that

$$(2.5) \quad \begin{cases} T(\mathbf{A}_1 + \mathbf{A}_2) = T(\mathbf{A}_1) + T(\mathbf{A}_2) \\ T(\mathbf{A}_1 \times \mathbf{A}_2) = T(\mathbf{A}_1) \times T(\mathbf{A}_2) \end{cases} \quad \forall \mathbf{A}_1, \mathbf{A}_2 \in \mathbb{C}^{n \times n}.$$

We also note that the matrix $\sum_{0 \leq k < n} \mathbf{B}(k, k)$ is unitary and corresponds to the Discrete Fourier Transform (DFT) matrix. from which it follows that the matrix

product of the DFT matrix with an arbitrary matrix \mathbf{A} with entries in $\mathbb{Q}\left[e^{i\frac{2\pi}{3}}\right]$ can be recovered without using complex numbers at all.

Incidentally there are families of complete orthogonal matrix basis analogous to the fourier basis associated with arbitrary unitary matrices and expressed by

$$(2.6) \quad \left\{ \mathbf{Q}_{\mathbf{U}}(k, l) = \sum_{0 \leq j < n} \left(\mathbf{e}_j \mathbf{e}_{\{j+k \bmod n\}}^t \right) u_{jl} \right\}_{0 \leq k, l < n}$$

where

$$\mathbf{U} \mathbf{U}^\dagger = \mathbf{I}$$

It also follows from this observation that for a hadamard matrix \mathbf{H} , the orthonormal basis

$$(2.7) \quad \left\{ \mathbf{Q}_{\mathbf{H}}(k, l) = \sum_{0 \leq j < n} \left(\mathbf{e}_j \mathbf{e}_{\{j+k \bmod n\}}^t \right) h_{jl} \right\}_{0 \leq k, l < n}$$

is a 2-pseudo-closed complete orthogonal basis spanning the set of $n \times n$ matrices.

Theorem 2. *Let M_n denote the vector space of $n \times n$ matrices with complex entries there exist a set of 2-pseudo-close complete orthogonal basis of full rank matrices \mathcal{B} which spans M_n if and only if*

$$(2.8) \quad \exists \mathbf{H} \in M_n \text{ such that } \mathbf{H}^T \mathbf{H} = \mathbf{I} \text{ and } \mathbf{H} \star \mathbf{H} = \mathbf{1}_{n \times n}.$$

Similarly to the fourier basis case, the complete orthogonal matrix basis associated with a hadamard matrix \mathbf{H} generate a finite matrix group \mathcal{H} of order bouded by $n2^n$ which we call *the Hadamard group*. The Hadamard group is isomorphic to a subgroup of S_{2n} and the group isomorphism R is described bellow

Corollary 3. *It follows that hadamard matrix \mathbf{H} induces a non trivial injective map*

$$R: M_n \rightarrow M_{2n}$$

$$\forall \mathbf{A} \in \mathbb{C}^{n \times n},$$

(2.9)

$$R(\mathbf{A}) = \sum_{0 \leq k, l < n} n^{-1} \langle \mathbf{A}, \mathbf{Q}_{\mathbf{H}}(k, l) \rangle \sum_{0 \leq j < n} \left(\mathbf{e}_j \mathbf{e}_{\{j+k \bmod n\}}^t \right) \otimes \begin{cases} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} & \text{if } h_{jl} = 1 \\ \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} & \text{if } h_{jl} = -1 \end{cases}.$$

so that

$$(2.10) \quad \begin{cases} R(\mathbf{A} + \mathbf{B}) = R(\mathbf{A}) + R(\mathbf{B}) \\ R(\mathbf{A} \times \mathbf{B}) = R(\mathbf{A}) \times R(\mathbf{B}) \end{cases}.$$

Proof. The fact that Hadamard matrices can be used to construct 2-pseudo-close complete orthogonal basis is immediate from the discussion in section[1], consequently our proof shall focus on showing that the existence of a 2-pseudo-close complete orthogonal basis of full rank matrices $\mathcal{B} = \{\mathbf{N}(k, l)\}_{0 \leq k, l < n}$, implies the existence of an $n \times n$ hadamard matrix.

If \mathcal{B} is a complete basis then it must also express diagonal matrices, hence for a diagonal matrix \mathbf{D} we have

$$(2.11) \quad \mathbf{D} = \sum_{0 \leq k, l < n} \frac{1}{\|\mathbf{N}(k, l)\|_{\ell_2}^2} \langle \mathbf{D}, \mathbf{N}(k, l) \rangle \mathbf{N}(k, l).$$

We recall that the defining property for diagonal matrices is the fact that

$$(2.12) \quad \forall m > 2 \quad \mathbf{D}^m = \mathbf{D}^{*^m}$$

from which it follows that the elements of \mathcal{B} for which $\langle \mathbf{D}, \mathbf{N}(k, l) \rangle \neq 0$ must also be diagonal matrices. Furthermore since the basis element have entries belong to the set $\{0, \pm 1\}$ and most importantly are full rank and it follows that the diagonal entries should be non zero and the diagonal elements of \mathcal{B} should span a vector space of dimension n , which completes the proof \square .

3. CONCLUSION

We have discussed here a variety of matrix bases and illustrated how the notion of k -pseudo closure for complete orthogonal matrix bases ties together closure properties with respect to the matrix product operation so often associated with groups on one hand and complete orthogonal basis commonly associated with vector spaces on the other hand. We point out that the k -pseudo-close complete matrix basis generalize Pauli matrices and simultaneously provide us with an alternative approach to generalizing the algebra of quaternions and octonions. Furthermore by analogy to discrete fourier analysis we argue that these basis suggest a natural framework for matrix fourier transform providing us with a choice of basis from which on might select the one which is best suited to some particular application. Finally from an algorithmic point of view, recalling the fact that the fast fourier transform plays a crucial role for fast integer mulitplication algorithm, It might be of interest to investigate whether matrix fourier transform and their corresponding convolution products also suggest efficient algorithms for matrix multiplication.

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REFERENCES

- [1] J. Patera and H. Zassenhaus, The Pauli Matrices in n dimensions grading of simple Lie Algebras of type A_{n-1} , AIP Journal of Mathematical Physics, 1987
- [2] Antonio Lao, Hadamard vs. Pauli-Dirac Matrices
www.toequest.com/forum/attachment.php?attachmentid=1589.
- [3] Kazuyuki Fujii, Quantum Optical Construction of Generalized Pauli and Walsh-Hadamard Matrices in Three Level Systems, *arXiv:quant-ph/0309132v1*
- [4] Maurice R Kibler, Variations on a theme of Heisenberg, Pauli and Weyl (2008) J. Phys. A: Math. Theor. 41 375302.

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